Exactification of Complex Projective Packings

Thomas Emerick, Murtadha Aljanabi

Louisiana State University

temeri2@lsu.edu

Abstract.

Configurations of lines in $\mathbb{C}^d$ with small pairwise inner products are closely related to equiangular tight frames (ETFs) and, in the maximal case, to symmetric informationally complete positive operator-valued measures (SIC-POVMs). Many best-known complex projective packings are available only as floating-point data, for example in the *Game of Sloanes* database [1]. Converting such numerical configurations into exact algebraic data, or *exactification*, permits symbolic verification and makes their arithmetic structure accessible.

We describe an exactification pipeline based on high-precision numerical approximation, lattice basis reduction, and minimal-polynomial recognition. Rather than exactifying frame coordinates directly, we work with triple products of the Gram matrix, which are invariant under the natural projective-unitary symmetries.

We apply this method to numerically obtained candidate minimizers of the 4-frame potential in parameter regimes where no ETF exists. For $(4, 6)$ and $(3, 5)$, we recover exact algebraic descriptions of the associated triple products. In the $(4, 6)$ case, we additionally obtain an exact expression for the 4-frame potential and an algebraic equation satisfied by the coherence. These calculations provide exact models for the numerical configurations, although rigorous global optimality remains open.

We also study several structured ETFs. For the $(4, 13)$ harmonic ETF arising from the $(13, 4, 1)$ cyclic difference set, we reconstruct the frame algebraically and determine its triple products symbolically. For the Paley ETF family [2], where $n = 2d + 1$ is prime and $n \equiv 3 \pmod 4$, we obtain closed-form triple products in $\mathbb{Q}(\sqrt{-n})$. Combined with a quadratic-residue classification of their index patterns [3], these formulas permit exact database files to be generated using streaming modular arithmetic without storing the full cubic tensor. The method produces exact files for examples including the $(15, 31)$ Paley ETF and scales to substantially larger parameters.

The talk will discuss the exactification pipeline, the distinction between exact algebraic results and numerical evidence, and the role of projective packings and frame invariants in quantum information.

Keywords: equiangular tight frames; projective line packings; exactification; triple products; quantum information

References

[1] Jasper, J., King, E. J., Mixon, D. G.: Game of Sloanes: Best known packings in complex projective space. arXiv:1907.07848 (2019).

[2] Renes, J. M.: Equiangular tight frames from Paley tournaments. Linear Algebra Appl. 426(2–3), 497–501 (2007).

[3] King, E. J.: k-Homogeneous equiangular tight frames. arXiv:2505.00160 (2025).

Acknowledgment.

The authors thank Dr. Gene Kopp and Dr. Joseph DiCapua for their instruction and guidance throughout the project.

The presenting author gratefully acknowledges the LSU Office of Undergraduate Research and the LSU Ogden Honors College for providing financial support to attend and present at the 10th Iberian Mathematical Meeting.